

Subprincipal symbol for Toeplitz operators

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Abstract

We establish some subprincipal estimates for Berezin-Toeplitz operators on symplectic compact manifolds. From this, we construct a family of subprincipal symbol maps and we prove that these maps are the only ones satisfying some expected conditions.

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Toeplitz operators on symplectic manifolds are similar to semiclassical pseudo-differential operators on cotangent bundles. In particular, we can extend to Toeplitz operators the usual techniques to describe the spectrum of pseudo-differential operators, as for instance the trace formula [BPU98] or the Bohr-Sommerfeld conditions [Cha03b], [Cha06], [Le 13]. Two important ingredients in these semi-classical results are the principal and subprincipal symbols of an operator. One issue is that there is no obvious definition for the subprincipal symbol of a Toeplitz operators, cf. as instance [BPU98] or [BdM02].

In this paper, we introduce some axioms that a subprincipal symbol should satisfy to our point of view. Then we construct all the subprincipal symbol satisfying these axioms. We work in the general setting introduced in [Cha15] for the quantization of symplectic manifolds. General reference for the Toeplitz operators are [BdMG81], [MM07], [BMS94], [Gui95]. Our construction is inspired from our previous work on Kähler manifolds, in particular [Cha06] and [Cha07], and uses in an essential way the metaplectic correction.

1 Statement of the results

To start with, consider a compact Kähler manifold M equipped with a holomorphic Hermitian line bundle L and with a holomorphic Hermitian vector bundle A . Assume that L is positive so that the Chern curvature of L is $\frac{1}{i}\omega$ where $\omega \in \Omega^2(M, \mathbb{R})$ is symplectic. For any $k \in \mathbb{N}$, let \mathcal{H}_k be the space of holomorphic sections of $L^k \otimes A$.

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More generally, let (M, ω) be any symplectic compact manifold such that $\frac{1}{2\pi}[\omega]$ is integral. Introduce a Hermitian line bundle $L \rightarrow M$ with a Hermitian connection ∇ of curvature $\frac{1}{i}\omega$, a Hermitian vector bundle A and an almost complex structure j compatible with ω . Then consider a family of finite dimensional subspace

$$\mathcal{H}_k \subset \mathcal{C}^\infty(M, L^k \otimes A), \quad k \in \mathbb{N}^*$$

consisting of almost-holomorphic sections in the sense of [Cha15]. In the sequel, we will refer to the case where M is Kähler, L and A holomorphic, ∇ is the Chern connection and \mathcal{H}_k consists of holomorphic sections, as the Kähler case. Observe that \mathcal{H}_k has a natural scalar product obtained by integrating the pointwise scalar product against the Liouville measure $\omega^n/n!$.

To these data is associated a star-algebra \mathcal{T} consisting of the so-called Berezin-Toeplitz operators or more briefly Toeplitz operators. The definition will be recalled later, let us just say for now that a Toeplitz operator T is an endomorphism family $(T_k : \mathcal{H}_k \rightarrow \mathcal{H}_k, k \in \mathbb{N})$. The product of \mathcal{T} is the usual composition of endomorphisms and the involution is the Hermitian adjoint. A very important fact is the existence of a natural star-algebra morphism $\sigma_p : \mathcal{T} \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ which is onto and with kernel

$$k^{-1}\mathcal{T} := \{(k^{-1}T_k)/(T_k) \in \mathcal{T}\} = \{(T_k) \in \mathcal{T} / \|T_k\| = \mathcal{O}(k^{-1})\}.$$

For any $T \in \mathcal{T}$, $\sigma_p(T)$ is called the principal symbol of T . Furthermore, if T and S are two Toeplitz operators with scalar valued principal symbol f and g respectively, then $ik[T, S]$ is a Toeplitz operator with principal symbol $\{f, g\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket corresponding to ω .

Let \mathcal{T}_{sc} be the subalgebra of \mathcal{T} consisting of operators with a scalar valued principal symbol. We say that a linear map $\sigma_s : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ is a *subprincipal symbol map* if it satisfies the following conditions:

- i) for any $T \in \mathcal{T}$, $\sigma_s(k^{-1}T) = \sigma_p(T)$,
- ii) for any $T \in \mathcal{T}_{\text{sc}}$, $\sigma_s(T^*) = \sigma_s(T)^*$,
- iii) for any $T, S \in \mathcal{T}_{\text{sc}}$, $\sigma_s(TS) = \sigma_p(T)\sigma_s(S) + \sigma_s(T)\sigma_p(S) + \frac{1}{2i}\{\sigma_p(T), \sigma_p(S)\}$,

With such a map σ_s , we can control the Toeplitz operators with a scalar principal symbol up to $k^{-2}\mathcal{T}$. More precisely, by i) and ii), the map

$$\sigma := \sigma_p + \hbar \sigma_s : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{C}^\infty(M) \oplus \hbar \mathcal{C}^\infty(M, \text{End } A)$$

is real, onto with kernel $k^{-2}\mathcal{T}_{\text{sc}}$.

As a first attempt to construct a subprincipal symbol map, we can consider the contravariant symbols. Let Π_k be the orthogonal projector of $\mathcal{C}^\infty(M, L^k \otimes A)$ onto \mathcal{H}_k . Recall that for any $f \in \mathcal{C}^\infty(M, \text{End } A)$, $(\Pi_k f, k \in \mathbb{N})$ is a Toeplitz operator with principal symbol f . So by the properties of the principal symbol recalled above, for any Toeplitz operator $T \in \mathcal{T}$, we have $T = \Pi f \mod k^{-1}\mathcal{T}$ with $f = \sigma_p(T)$. We can now set $\sigma_s^c(T) := \sigma_p(k(T - \Pi f))$, so that we have

$$T = \Pi f + k^{-1}\Pi g \mod k^{-2}\mathcal{T} \quad \text{with } f = \sigma_p(T), \quad g = \sigma_s^c(T) \quad (1)$$

This defines a map σ_s^c which satisfies i), ii) but not iii). For instance, in the Kähler case with A the trivial line bundle, we proved in [Cha03a] that $\sigma_s^c(ST) = \sigma_p(T)\sigma_s^c(S) + \sigma_s^c(T)\sigma_p(S) + \hbar B(\sigma(S), \sigma(T))$ where in complex coordinates (z^i) ,

$$B(f, g) = - \sum G^{jk} \partial_{z_j} f \cdot \partial_{\bar{z}_k} g \quad \text{if} \quad \omega = i \sum G_{j,k} dz_j \wedge d\bar{z}_k.$$

As expected, the antisymmetric part of B is $\frac{1}{2i}$ times the Poisson bracket. But the symmetric part does not vanish. Nevertheless, setting

$$\sigma_s(T) = \sigma_s^c(T) + \frac{\hbar}{2} \Delta \sigma_p(T), \quad (2)$$

where $\Delta = \sum G^{jk} \partial_{z_j} \partial_{\bar{z}_k}$, we obtain a subprincipal symbol map, which was introduced in [Cha03b] to state the Bohr-Sommerfeld conditions. Equation (2) may be viewed as a generalization to Kähler manifolds of the formula giving the subprincipal term of the Weyl symbol in terms of the anti-Wick symbol. Still in the Kähler case, this approach can be generalized to any holomorphic vector bundle A by using the expression for B obtained in [MM12]. In the general symplectic case, we only know that B is a bidifferential operator. Using some standard argument in deformation quantization, this is enough to deduce the existence of a subprincipal symbol map, cf. Proposition 3.2. But we do not have an explicit formula defining this symbol.

Actually, there is a direct way to define an explicit subprincipal symbol. The construction is easier in the case where the canonical bundle $K = \wedge^{n,0} T^*M$ has a square root (δ, φ) , what we assume from now on. Here δ is a Hermitian line bundle over M and φ is an isomorphism from δ^2 to K . It is known that such a square root exists if and only if M has a spin structure if and only if the second Stiefel-Whitney class of M vanishes.

Set $B := A \otimes \delta^{-1}$ so that $A = B \otimes \delta$. Choose a Hermitian connection ∇^B of B . Denote by ∇^k the connection of $L^k \otimes B$ induced by ∇ and ∇^B . For any vector field X of M , let D_X^K be the derivative of $\mathcal{C}^\infty(M, K)$ in the direction of X given by

$$D_X^K(s) = p(\mathcal{L}_X s), \quad \forall s \in \mathcal{C}^\infty(M, K),$$

where p is the projection from $\wedge^n(T^*M \otimes \mathbb{C})$ onto K with kernel $\wedge^{n-1,1} T^*M \oplus \dots \oplus \wedge^{0,n} T^*M$. Introduce now the derivative D_X^δ of $\mathcal{C}^\infty(M, \delta)$ in the direction of X such that

$$D_X^K(\varphi(s^2)) = 2\varphi(s \otimes D_X^\delta s), \quad \forall s \in \mathcal{C}^\infty(M, \delta).$$

Finally for any $f \in \mathcal{C}^\infty(M)$, consider the operator

$$Q_k(f) = \Pi_k \left(f + \frac{i}{\hbar} (\nabla_X^k \otimes \text{id} + \text{id} \otimes D_X^\delta) \right) : \mathcal{H}_k \rightarrow \mathcal{H}_k.$$

where X is the Hamiltonian vector field of f , that is $df = \omega(X, \cdot)$.

Theorem 1.1. *For any Hermitian connection ∇^B of B , we have*

1. *For any $f \in \mathcal{C}^\infty(M)$, $Q(f) = (Q_k(f))$ is a Toeplitz operator with principal symbol f .*

2. For any f and $g \in \mathcal{C}^\infty(M)$, we have

$$Q_k(f)Q_k(g) = Q_k(fg) + \frac{1}{2ik}Q_k(\{f, g\}) \mod k^{-2}\mathcal{T}$$

and

$$[Q_k(f), Q_k(g)] = \frac{1}{ik}Q_k(\{f, g\}) + \frac{1}{ik^2}\Pi_k(R(X, Y)) \mod k^{-3}\mathcal{T}, \quad (3)$$

where X and Y are the Hamiltonian vector fields of f and g respectively, $iR = \Theta(B) \in \Omega^2(M, \text{End } B)$ is the curvature of ∇^B .

3. For any $f \in \mathcal{C}^\infty(M)$ and Toeplitz operator S , $(ki[Q_k(f), S])$ is a Toeplitz operator with principal symbol

$$\sigma_p(ki[Q_k(f), S]) = -\nabla_X^B(\sigma_p(S)). \quad (4)$$

where X is the Hamiltonian vector field of f .

In this statement, we have identified $\text{End } A$ and $\text{End } B$ by using that δ is a line bundle so that $\text{End}(\delta)$ is canonically isomorphic with the trivial line bundle. Furthermore, we let the covariant derivative ∇^B act on sections of $\text{End } B$ in the usual way. Observe that in the case where B is a line bundle $\nabla_X^B(\sigma_p(S)) = X \cdot \sigma_p(S)$ so that Equation (4) becomes $\sigma_p(ki[Q_k(f), S]) = \{f, \sigma_p(S)\}$.

We can now define a subprincipal symbol map $\sigma_s : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ by $\sigma_s(T) := \sigma_p(k(T - Q(\sigma_p(T))))$. In other words, for any Toeplitz operator $T \in \mathcal{T}_{\text{sc}}$, we have

$$T = Q(f) + k^{-1}\Pi g \mod k^{-2}\mathcal{T}, \quad \text{with } f = \sigma_p(T), \ g = \sigma_s(T) \quad (5)$$

By Theorem 1.1, σ_s satisfies i), ii) and iii). Furthermore, by (3) and (4), for any $S, T \in \mathcal{T}_{\text{sc}}$,

$$\sigma_s(ik[T, S]) = R(X, Y) - \nabla_X^B \sigma_s(S) + \nabla_Y^B \sigma_s(T) + i[\sigma_s(T), \sigma_s(S)] \quad (6)$$

where X and Y are the Hamiltonian vector fields of $\sigma_p(T)$ and $\sigma_p(S)$ respectively.

Observe that σ_s is not uniquely defined, since it depends on the choice of ∇^B . Actually, the space of Hermitian connections of B is an affine space directed by $\Omega^1(M, \text{Herm } B)$ where $\text{Herm } B$ is the bundle of Hermitian endomorphisms of B . As we will see, the space of subprincipal symbol maps is an affine space directed by $\mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$, the action being given by $(V + \sigma_s)(T) = \sigma_s(T) + V \cdot \sigma_p(T)$.

Proposition 1.2. *The map from the space of connections of B to the space of subprincipal symbol maps of \mathcal{T} , sending ∇^B to σ_s so that (5) is satisfied, is an isomorphism of affine vector spaces. Here the corresponding vector spaces $\Omega^1(M, \text{Herm } B)$ and $\mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$ are identified through the symplectic duality $T^*M \simeq TM$ and the natural isomorphism $\text{Herm } A \simeq \text{Herm } B \otimes \text{Herm } \delta \simeq \text{Herm } B$ considered above.*

Remark 1.3. The case where B is trivial, so that $\mathcal{H}_k \subset \mathcal{C}^\infty(M, L^k \otimes \delta)$, is usually called the quantization with metaplectic correction. There is a natural and very particular choice for the connection of B : $\nabla^B = d$. Then (3) simplifies

$$[Q_k(f), Q_k(g)] = \frac{1}{ik} Q_k(\{f, g\}) \mod k^{-3} \mathcal{T}$$

and the corresponding subprincipal symbol satisfies:

$$\sigma_s(ik[T, S]) = \{\sigma_p(T), \sigma_s(S)\} + \{\sigma_s(T), \sigma_p(S)\}. \quad (7)$$

This situation can be compared with the one of pseudo-differential operators acting on half-densities. In that case, there is a well-defined subprincipal symbol which satisfies (7). Let us note also that according to [BdM02], only a map satisfying Equation (7) deserves the name of subprincipal symbol. \square

Remark 1.4. Equation (3) is relevant to classify the algebras \mathcal{T} corresponding to various choices of A . For instance, assume that A is a line bundle so that $R \in \Omega^2(M, \mathbb{R})$. Then the cohomology class of R does not depend on the choice of the subprincipal symbol. Actually, iR being the curvature of ∇^B ,

$$\frac{1}{2\pi}[R] = c_1(B) = c_1(A) + \frac{1}{2}c_1(M).$$

Assume now that we have another line bundle A' on M with corresponding spaces $\mathcal{H}'_k \subset \mathcal{C}^\infty(M, L^k \otimes A')$ and Toeplitz algebra \mathcal{T}' . We may ask whether there exists a star-algebra morphism $\Phi : \mathcal{T} \rightarrow \mathcal{T}'$ such that $\Phi(k^{-1}T) = k^{-1}\Phi(T)$ and $\sigma_p(\Phi(T)) = \sigma_p(T)$ for any $T \in \mathcal{T}$. Observe that if Φ is such a morphism and σ'_s is a subprincipal symbol map of \mathcal{T}' , then $\sigma_s := \sigma'_s \circ \Phi$ is a subprincipal symbol map of \mathcal{T} . So as a consequence of Equation (3), a necessary condition for Φ to exist is that A and A' have the same Chern class. In [Cha07], we prove that this condition is also sufficient in the Kähler case. We plan to extend these results to the general symplectic case in a next paper. \square

Remark 1.5. Recall that the star products of (M, ω) are classified up to equivalence by their Fedosov class which is an element of $H^2(M, \mathbb{C})[[\hbar]]$, cf. [Fed96]. In the case where A is a line bundle, the product \star_{cont} of contravariant symbols of Toeplitz operators is a star product. By Equation (3), the first coefficient of the Fedosov class of \star_{cont} is equal to $c_1(A) + \frac{1}{2}c_1(M)$. \square

Remark 1.6. In the Kähler case, there is a particular choice for the connection of B , namely the Chern connection. Doing this choice and assuming that A is a line bundle, we recover the subprincipal symbol defined in equation (2), cf. [Cha06]. In this case, Equations (2) and (3) have been proved in [Cha07]. However, the proof in [Cha07] was rather indirect and based on the morphisms which we alluded to in Remark 1.4. The proof we propose in this paper is much simpler. \square

Theorem 1.1 and Proposition 1.2 generalize to the case where there is no half-form bundle. The difficulty as we will see is to define the convenient operators

$Q_k(f)$ in this case and to understand what replaces the choice of the connection on B .

The paper is organized as follows. In Section 2, we recall some basic facts on Toeplitz operators. In Section 3, we start the study of subprincipal symbols and we go as far as possible without using the operators $Q_k(f)$. We will see that we can deduce Equation (6) from almost nothing, but without computing explicitly R and ∇^B . In section 4, we introduce some material related to half-form bundles. In section 5, we define the operator $Q_k(f)$ and state the theorem generalizing Theorem 1.1 in the absence of half-form bundle. Sections 6 and 7 are devoted to the proof of this result.

2 Toeplitz operators

Consider a compact symplectic manifold M equipped with a prequantum bundle $L \rightarrow M$. Recall that L is a Hermitian line bundle with a connection of curvature $\frac{1}{i}\omega$. Let A be any Hermitian vector bundle. Let j be an almost complex structure of M compatible with ω . Consider a family

$$\mathcal{H} = (\mathcal{H}_k \subset \mathcal{C}^\infty(M, A \otimes L^k), k \in \mathbb{N})$$

of finite dimensional subspaces. Assume that the orthogonal projector Π_k onto \mathcal{H}_k belongs to the algebra \mathcal{A}_0 introduced in Section 6.

A *Toeplitz operator* is any family $(T_k : \mathcal{H}_k \rightarrow \mathcal{H}_k, k \in \mathbb{N})$ of operators of the form

$$T_k = \Pi_k f(\cdot, k) + R_k, \quad k \in \mathbb{N}^* \quad (8)$$

where $f(\cdot, k)$, viewed as a multiplication operator, is a sequence in $\mathcal{C}^\infty(M, \text{End } A)$ admitting an asymptotic expansion $f_0 + k^{-1}f_1 + \dots$ for the \mathcal{C}^∞ topology. Furthermore the norm of $R_k \in \text{End } \mathcal{H}_k$ is in $\mathcal{O}(k^{-\infty}) = \bigcap_N \mathcal{O}(k^{-N})$.

The following facts are proved in [Cha15]. The space \mathcal{T} of Toeplitz operators is a star-algebra with identity (Π_k) , the product being the usual composition of operators, the involution being the Hermitian adjoint. The symbol map

$$\sigma_{\text{cont}} : \mathcal{T} \rightarrow \mathcal{C}^\infty(M, \text{End } A)[[\hbar]]$$

sending (T_k) into the formal series $f_0 + \hbar f_1 + \dots$ where the functions f_i are the coefficients of the asymptotic expansion of the multiplier $f(\cdot, k)$ is well defined. It is onto and its kernel is the ideal consisting of $\mathcal{O}(k^{-\infty})$ Toeplitz operators. More precisely, for any integer ℓ , $\|T_k\| = \mathcal{O}(k^{-\ell})$ if and only if $\sigma_{\text{cont}}(T) = \mathcal{O}(\hbar^\ell)$. According to Berezin terminology, $\sigma_{\text{cont}}(T)$ is called the *contravariant* symbol of T .

The *principal* symbol $\sigma_p(T) \in \mathcal{C}^\infty(M, \text{End } A)$ is by definition the first coefficient of the contravariant symbol, so $\sigma_{\text{cont}}(T) = \sigma_p(T) + \mathcal{O}(\hbar)$. The principal symbol map

$$\sigma_p : \mathcal{T} \rightarrow \mathcal{C}^\infty(M, \text{End } A)$$

is onto and $\sigma_p(T) = 0$ if and only if there exists $S \in \mathcal{T}$ such that $T = k^{-1}S$. For any Toeplitz operators $T, S \in \mathcal{T}$ with principal symbols f and g , we have $\sigma_p(TS) = f \cdot g$. If f and g are scalar valued, then $ik[T, S] \in \mathcal{T}$ and

$$\sigma_p(ik[T, S]) = \{f, g\}, \quad (9)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of M . Denoting by $\|T_k\|$ the operator norm of T_k corresponding to the scalar product of $\mathcal{H}_k \subset C^\infty(M, L^k \otimes A)$, we have

$$\|T_k\| = \sup_{y \in M} |\sigma_p(T)(y)| + \mathcal{O}(k^{-1})$$

where for any $y \in M$, $|\sigma_p(T)(y)|$ is the operator norm of $\sigma_p(T)(y) \in \text{End } A_y$. Consequently, $\sigma_p(T) = 0$ if and only if $\|T_k\| = \mathcal{O}(k^{-1})$.

As a last property, the full product of contravariant symbols has the following form: if $\sigma_{\text{cont}}(T) = \sum \hbar^\ell f_\ell$ and $\sigma_{\text{cont}}(S) = \sum \hbar^\ell g_\ell$, then

$$\sigma_{\text{cont}}(TS) = \sum_{\ell} \hbar^\ell \sum_{\ell=p+q+r} B_r(f_p, g_q) \quad (10)$$

where $B_0(f, g) = fg$ and for any r , $B_r : C^\infty(M, \text{End } A) \times C^\infty(M, \text{End } A) \rightarrow C^\infty(M, \text{End } A)$ is a bilinear local operator, cf. [Cha15].

3 Subprincipal symbols

Denote by \mathcal{T}_{sc} the set of Toeplitz operator $T \in \mathcal{T}$ with a scalar principal symbol. Let \mathcal{E} be the set of linear map $\sigma_s : \mathcal{T}_{\text{sc}} \rightarrow C^\infty(M, \text{End } A)$ satisfying the following conditions:

- i) for any $T \in \mathcal{T}$, $\sigma_s(k^{-1}T) = \sigma_p(T)$,
- ii) for any $T \in \mathcal{T}_{\text{sc}}$, $\sigma_s(T^*) = \sigma_s(T)^*$,
- iii) for any $T, S \in \mathcal{T}_{\text{sc}}$, $\sigma_s(TS) = \sigma_p(T)\sigma_s(S) + \sigma_s(T)\sigma_p(S) + \frac{1}{2i}\{\sigma_p(T), \sigma_p(S)\}$,

We will first prove that \mathcal{E} is not empty. Define the contravariant subprincipal symbol map $\sigma_s^c : \mathcal{T}_{\text{sc}} \rightarrow C^\infty(M, \text{End } A)$ as follows:

$$\sigma_{\text{cont}}(T) = \sigma_p(T) + \hbar \sigma_s^c(T) + \mathcal{O}(\hbar^2) \quad (11)$$

Then σ_s^c satisfies i), ii) but not iii). Actually, by (10), we have

$$\sigma_s^c(TS) = \sigma_p(T)\sigma_s^c(S) + \sigma_s^c(T)\sigma_p(S) + B(\sigma_p(T), \sigma_p(S)) \quad (12)$$

where B is a bidifferential operator from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M, \text{End } A)$. By (9), the antisymmetric part of B is equal to $\frac{1}{2i}$ times the Poisson bracket, so

$$B(f, g) = B^s(f, g) + \frac{1}{2i}\{f, g\} \quad (13)$$

where B^s is symmetric. We will modify σ_s^c to get a subprincipal symbol satisfying iii). The method we follow is based on standard lemmas in Deformation quantization.

Proposition 3.1. *Let $B^s : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ be any bidifferential symmetric operator satisfying*

$$B^s(f, g)h + B^s(fg, h) = B^s(f, gh) + fB^s(g, h), \quad \forall f, g, h \in \mathcal{C}^\infty(M). \quad (14)$$

Then there exists a differential operator $Q : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ such that for any $f, g \in \mathcal{C}^\infty(M)$, we have $B^s(f, g) = fQ(g) + Q(f)g - Q(fg)$.

In the case where A is the trivial line bundle over M , this result is a classical lemma in Hochschild cohomology saying that any cocycle with a null antisymmetric part is exact, cf. [BCG97] for a short proof. It is easy to deduce Proposition 3.1 from this particular case.

Using that the product of Toeplitz operators is associative, a straightforward computations shows that the bidifferential operator B defined by (12) satisfies the equality $B(f, g)h + B(fg, h) = B(f, gh) + fB(g, h)$. Using that the Poisson bracket is a derivative with respect to each of his arguments, we conclude that the symmetric part B^s of B satisfies (14). Applying Proposition 3.1, we get a differential operator Q . Then a straightforward computation shows that $\sigma_s := \sigma_s^c + Q \circ \sigma_p$ satisfies iii). Since σ_s^c satisfies ii), B^s is real in the sense that B^s is the complexification of a \mathbb{R} -bilinear map $\mathcal{C}^\infty(M, \mathbb{R}) \times \mathcal{C}^\infty(M, \mathbb{R}) \rightarrow \mathcal{C}^\infty(M, \text{Herm } A)$, where $\text{Herm } A \rightarrow M$ is the vector bundle of Hermitian endomorphisms of A . B^s being real, we can choose Q real, so that σ_s^c satisfies ii) as well. We have proved that \mathcal{E} is not empty.

Now consider $\sigma_s \in \mathcal{E}$ and $V \in \mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$. Then the map σ'_s defined by

$$\sigma'_s(T) := \sigma_s(T) + df(V) \quad \text{where} \quad f = \sigma_p(T),$$

satisfies i), ii), iii). Conversely, let σ_s and σ'_s in \mathcal{E} . By i), $\sigma'_s - \sigma_s$ vanishes on $k^{-1}\mathcal{T}$. Since $k^{-1}\mathcal{T}$ is the kernel of the principal symbol map, we have $\sigma'_s(T) - \sigma_s(T) = D(\sigma_p(T))$ where $D : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M, \text{End } A)$. By ii), D is real. By iii), D is a derivation, that is $D(fg) = fD(g) + gD(f)$. We conclude that $D(f) = df(V)$ for some $V \in \mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$. To summarize we have proved the following Proposition.

Proposition 3.2. *\mathcal{E} is an affine space with associated vector space $\mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$.*

It is helpful to view $\sigma_s \in \mathcal{E}$ as a first order deformation of the principal symbol. To give a sense to this, denote by \mathcal{S} be the vector space $\mathcal{C}^\infty(M) \oplus \hbar\mathcal{C}^\infty(M, \text{End } A)$ and define the map $\sigma : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{S}$ by

$$\sigma(T) = \sigma_p(T) + \hbar\sigma_s(T).$$

Using that σ_p is onto with kernel $k^{-1}\mathcal{T}$ and that σ_s satisfies i), we see that σ is onto with kernel $k^{-2}\mathcal{T}$. By ii), σ is real. Furthermore by iii), $\sigma(ST) = \sigma(S) \star \sigma(T)$ where \star is the product of \mathcal{S} given by

$$(f_0 + \hbar f_1) \star (g_0 + \hbar g_1) = f_0 g_0 + \hbar(f_0 g_1 + f_1 g_0 + \frac{1}{2i}\{f_0, g_0\}).$$

1 being the identity of \star , we easily get that $\sigma(\text{id}) = 1$.

For any $T, S \in \mathcal{T}_{\text{sc}}$, $k[T, S]$ is a Toeplitz operator with scalar symbol. Furthermore observe that the class of $k[T, S]$ modulo $\mathcal{O}(k^{-2})$ only depend on the classes of T and S modulo $\mathcal{O}(k^{-2})$. So there exists a unique bilinear map

$$[\cdot, \cdot]_{\sigma} : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$$

such that $\sigma(ik[T, S]) = [\sigma(T), \sigma(S)]_{\sigma}$ for any $T, S \in \mathcal{T}_{\text{sc}}$. Since the commutator of endomorphisms is a derivation with respect to each argument and satisfies the Jacobi identity, we obtain that for any $f, g, h \in \mathcal{S}$

$$[f \star g, h]_{\sigma} = f \star [g, h]_{\sigma} + [f, h]_{\sigma} \star g. \quad (15)$$

and

$$[f, [g, h]_{\sigma}]_{\sigma} + [g, [h, f]_{\sigma}]_{\sigma} + [h, [f, g]_{\sigma}]_{\sigma} = 0 \quad (16)$$

Furthermore, $[f, g]_{\sigma} = -[g, f]_{\sigma}$ and $[f^*, g^*]_{\sigma} = [f, g]_{\sigma}^*$. Exploiting these equations, we deduce the following proposition.

Proposition 3.3. *For any $\sigma_s \in \mathcal{E}$, there exists $R \in \Omega^2(M, \text{Herm } A)$ and a connection $\nabla : \mathcal{C}^{\infty}(M, \text{Herm } A) \rightarrow \Omega^1(M, \text{Herm } A)$ such that for any $f_0 + \hbar f_1, g_0 + \hbar g_1 \in \mathcal{S}$ we have*

$$[f_0 + \hbar f_1, g_0 + \hbar g_1]_{\sigma} = \{f_0, g_0\} + \hbar(R(X, Y) - \nabla_X g_1 + \nabla_Y f_1 + i[f_1, g_1])$$

where X and Y are the Hamiltonian vector fields of f_0 and g_0 . Furthermore,

$$\text{coursb } \nabla = i \text{ ad}_R, \quad \nabla R = 0, \quad \nabla \text{id} = 0$$

and $\nabla(f_1, g_1) = (\nabla f_1) \cdot g_1 + f_1 \cdot (\nabla g_1)$ for any $f_1, g_1 \in \mathcal{C}^{\infty}(M, \text{Herm } A)$.

Proof. By condition i) and the properties of the principal symbol, we have

$$[f_0 + \hbar f_1, g_0 + \hbar g_1]_{\sigma} = \{f_0, g_0\} + \hbar(A(f_0, g_0) + B(f_0, g_1) + C(g_0, f_1) + i[f_1, g_1]),$$

where A, B, C are bilinear operators with value in $\mathcal{C}^{\infty}(M, \text{End } A)$, A being defined on $\mathcal{C}^{\infty}(M) \times \mathcal{C}^{\infty}(M)$ and B, C on $\mathcal{C}^{\infty}(M) \times \mathcal{C}^{\infty}(M, \text{End } A)$. Since $[\cdot, \cdot]_{\sigma}$ is antisymmetric, we have that

$$A(f_0, g_0) = -A(g_0, f_0), \quad B(f_0, g_1) = -C(f_0, g_1) \quad (17)$$

Since σ is real and $\sigma(\text{id}) = 1$, $[\cdot, \cdot]_{\sigma}$ is real and $[1, \cdot]_{\sigma} = 0$. Consequently, A and B are real, meaning that $A(\overline{f_0}, \overline{g_0}) = A(f_0, g_0)^*$ and $B(\overline{f_0}, g_1^*) = B(f_0, g_1)^*$. Furthermore

$$A(1, g_0) = 0, \quad B(1, g_1) = 0, \quad B(f_0, \text{id}) = 0. \quad (18)$$

The last equation follows from the fact that $\sigma(k^{-1} \text{id}) = \hbar \text{id}$ so that $[\hbar \text{id}, \cdot]_{\sigma} = 0$. By equation (15), for any $f_0, g_0 \in \mathcal{C}^{\infty}(M)$, and $h_1 \in \mathcal{C}^{\infty}(M, \text{End } A)$

$$[f_0 \star g_0, \hbar h_1]_{\sigma} = f_0 \star [g_0, \hbar h_1]_{\sigma} + [f_0, \hbar h_1]_{\sigma} \star g_0$$

so that

$$B(f_0 g_0, h_1) = f_0 B(g_0, h_1) + g_0 B(f_0, h_1) \quad (19)$$

Furthermore, as another application of Equation (15),

$$[f_0, g_0 \star \hbar h_1]_\sigma = [f_0, g_0]_\sigma \star \hbar h_1 + g_0 \star [f_0, \hbar h_1]_\sigma$$

so that

$$B(f_0, g_0 h_1) = \{f_0, g_0\} h_1 + g_0 B(f_0, h_1) = -(X \cdot g_0) h_1 + g_0 B(f_0, h_1). \quad (20)$$

where X is the Hamiltonian vector field of f_0 . By Equation (20), $B(f_0, \cdot)$ is a derivative of $\mathcal{C}^\infty(M, \text{End } A)$ in the direction of $-X$. By Equation (19) and the second equation of (18), for any $p \in M$, $B(f_0, g_1)(p) = B(f'_0, g_1)(p)$ if f_0 and f'_0 have the same differential at p . These two facts imply that $B(f_0, h_1) = -\nabla_X h_1$ for a connection $\nabla : \mathcal{C}^\infty(M, \text{End } A) \rightarrow \Omega^1(M, \text{End } A)$. Since B is real, ∇ is actually the complexification of a connection $\mathcal{C}^\infty(M, \text{Herm } A) \rightarrow \Omega^1(M, \text{Herm } A)$. Furthermore by the last equation of (18), $\nabla \text{id} = 0$.

Consider now $f, g, h \in \mathcal{C}^\infty(M)$. Expanding Equation (15) and using Jacobi identity for the Poisson bracket, we obtain that

$$A(fg, h) = fA(g, h) + gA(f, h).$$

Since furthermore A is antisymmetric and vanishes on the constant, we conclude that $A(f, g) = R(X, Y)$ with $R \in \Omega^2(M, \text{End } A)$ and X, Y the Hamiltonian vector fields of f and g . Since A is real, $R \in \Omega^2(M, \text{Herm } A)$.

Still working with $f, g, h \in \mathcal{C}^\infty(M)$ and denoting by X, Y and Z their Hamiltonian vector fields, we have

$$\begin{aligned} [f, [g, h]_\sigma]_\sigma &= \{f, \{g, h\}\} + \hbar(A(f, \{g, h\}) + B(f, A(g, h))) \\ &= \{f, \{g, h\}\} - \hbar(R(X, [Y, Z]) + \nabla_X R(Y, Z)) \end{aligned}$$

Since the bracket $[\cdot, \cdot]_\sigma$ satisfies the Jacobi identity, this shows that $(\nabla R)(X, Y, Z) = 0$ so that $\nabla R = 0$. Another application of the Jacobi identity (16) with $f, g \in \mathcal{C}^\infty(M)$ and $h \in \hbar \mathcal{C}^\infty(M, \text{End } A)$ leads to

$$[\nabla_X, \nabla_Y] h_1 - \nabla_{[X, Y]} h_1 = i[R(X, Y), h_1]$$

which means that the curvature of ∇ is $i \text{ad}_R$. The last equation to prove does not follow from the previous conditions. Consider three Toeplitz operators R, S and T with principal symbols $f_0 \in \mathcal{C}^\infty(M)$, $g_1, h_1 \in \mathcal{C}^\infty(M, \text{End } A)$. Starting from $ik[T, k^{-1}RS] = R(ik[T, k^{-1}S]) + (ik[T, k^{-1}R])S$, we obtain that

$$\nabla_X(g_1 \cdot h_1) = g_1 \cdot (\nabla_X h_1) + (\nabla_X g_1) \cdot h_1$$

where X is the Hamiltonian vector field of f_0 . □

We can also easily compute how the form R and the connection ∇ are changed when we choose a new subprincipal symbol.

Lemma 3.4. *Let $\sigma_s \in \mathcal{E}$, $V \in \mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$ and set $\sigma'_s = \sigma_s + V \cdot \sigma_p$. Then the connections ∇, ∇' and the two forms R, R' corresponding respectively to σ_s and σ'_s satisfy*

$$\nabla' = \nabla + i \text{ad}_\alpha, \quad R' = R + \nabla \alpha + i \alpha \wedge \alpha$$

where $\alpha \in \Omega^1(M, \text{Herm } A)$ is given by $\alpha = \omega(V, \cdot)$.

The case where A is a line bundle is already interesting. By the following corollary, the cohomology class $[R] \in H^2(M, \mathbb{R})$ does not depend on the choice of the subprincipal symbol map.

Corollary 3.5. *Assume that A is a line bundle. Then for any $\sigma_s \in \mathcal{E}$, we have*

$$[f_0 + \hbar f_1, g_0 + \hbar g_1]_\sigma = \{f_0, g_0\} + \hbar(R(X, Y) - \mathcal{L}_X g_1 + \mathcal{L}_Y f_1)$$

where $R \in \Omega^2(M, \mathbb{R})$ is closed and $\mathcal{L}_X, \mathcal{L}_Y$ are the Lie derivatives with respect to the Hamiltonian vector fields of f_0 and g_0 respectively. Furthermore, the two-form R' corresponding to $\sigma'_s = \sigma_s + V \cdot \sigma_p$ is given by $R' = R + d_V \omega$. So the cohomology class of R does not depend on the choice of σ_s .

Proof. Since A is a line bundle, $\text{Herm } A$ is naturally isomorphic with the trivial real line bundle. The fact that $\nabla \text{id} = 0$ implies that ∇ is the de Rham derivation. \square

At this point, we could believe that for any $\sigma_s \in \mathcal{E}$, there exists some connection $\nabla^A : \mathcal{C}^\infty(M, A) \rightarrow \Omega^1(M, A)$ preserving the metric of A , with curvature R and such that ∇ is the corresponding connection of $\text{Herm } A$. Indeed this would explain the equations in Proposition 3.3. We could also believe that the connection corresponding to $\sigma'_s = \sigma_s + V \cdot \sigma_p$ is given by $\nabla^A + \frac{1}{i} \iota_V \omega$ which would imply the equations in Lemma 3.4.

As we will see, this explanation almost holds, but we have to take into account the metaplectic correction. For instance, we will see that in the case where A is a line bundle, the cohomology class of $[\frac{1}{2\pi} R]$ is not $c_1(A)$ but $c_1(A) + \frac{1}{2} c_1(M)$.

4 Half-form computations

4.1 Canonical bundle and derivatives

Let $2n$ be the dimension of M . Let $K = \wedge^{n,0} T^* M$ be the canonical bundle of M with respect to j . Denote by p the projection $\wedge^n(T^* M \otimes \mathbb{C}) \rightarrow \wedge^{n,0} T^* M$ with kernel $\wedge^{n-1,1} T^* M \oplus \dots \oplus \wedge^{0,n} T^* M$. Then for any vector field X of M , introduce the derivative

$$D_X^K : \mathcal{C}^\infty(M, K) \rightarrow \mathcal{C}^\infty(M, K), \quad D_X^K \mu = p \mathcal{L}_X \mu \quad (21)$$

where \mathcal{L}_X is the Lie derivative with respect to X .

Lemma 4.1. *For any $X \in \mathcal{C}^\infty(M, TM)$ and $f \in \mathcal{C}^\infty(M)$ we have*

$$D_{fX}^K = fD_X^K + df(X^{1,0}),$$

where $X^{1,0} \in \mathcal{C}^\infty(M, T^{1,0}M)$ is the $(1,0)$ -component of X . Furthermore if X is symplectic, D_X^K preserves the metric of K induced by ω .

D_X^K preserves the metric means that for any sections $s, t \in \mathcal{C}^\infty(M, K)$ we have

$$\mathcal{L}_X(s, t) = (D_X^K s, t) + (s, D_X^K t)$$

where (s, t) is the pointwise scalar product. Equivalently we say that D_X^K is Hermitian.

Proof. By Cartan formula, we have $\mathcal{L}_{fX} - f\mathcal{L}_X = df \wedge \iota_X$. Introduce a local frame (∂_i) of $T^{1,0}M$ and denote by (θ_i) the dual frame. Since D_{fX}^K and fD_X^K are both derivatives in the direction of fX , they differ by the multiplication by a function. We compute this function by testing on the frame $\theta = \theta_1 \wedge \dots \wedge \theta_n$ of the canonical bundle. Write $X^{1,0} = \sum X_i \partial_i$ so that

$$\iota_X \theta = \sum (-1)^{i+1} X_i \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \theta_n$$

and

$$p(df \wedge \iota_X \theta) = \sum X_i df(\partial_i) \theta = df(X^{1,0}) \theta.$$

Consequently, $D_{fX}^K = fD_X^K + df(X^{1,0})$.

Let us prove that D_X^K preserves the metric of K when X is symplectic. Recall that for any sections s, t of K , $s \wedge \bar{t} = C_n(s, t) \omega^n$ for some constant C_n independent of s, t . Since

$$\mathcal{L}_X(s \wedge \bar{t}) = (\mathcal{L}_X s) \wedge \bar{t} + s \wedge \overline{\mathcal{L}_X t} = (D_X^K s) \wedge \bar{t} + s \wedge \overline{D_X^K t},$$

we deduce that $\mathcal{L}_X(s, t) = (D_X^K s, t) + (s, D_X^K t)$ when X satisfies $\mathcal{L}_X \omega^n = 0$. \square

Lemma 4.2. *For any vector fields X, Y of M , we have*

$$[D_X^K, D_Y^K] = D_{[X, Y]}^K + B_j(X, Y)$$

where $B_j(X, Y)$ is the function of M given by

$$B_j(X, Y) = \sum (\mathcal{L}_X \theta_i)(\bar{\partial}_j)(\mathcal{L}_Y \bar{\theta}_j)(\partial_i) - (\mathcal{L}_Y \theta_i)(\bar{\partial}_j)(\mathcal{L}_X \bar{\theta}_j)(\partial_i)$$

with (∂_i) a local frame of $T^{1,0}M$ and (θ_i) the dual frame.

Proof. Since

$$p\mathcal{L}_X p\mathcal{L}_Y p - p\mathcal{L}_X \mathcal{L}_Y p = p\mathcal{L}_X(p - \text{id})\mathcal{L}_Y p,$$

we have

$$[D_X^K, D_Y^K] - D_{[X, Y]}^K = p\mathcal{L}_X(p - \text{id})\mathcal{L}_Y - p\mathcal{L}_Y(p - \text{id})\mathcal{L}_X$$

Let $\theta = \theta_1 \wedge \dots \wedge \theta_n$. We have

$$\mathcal{L}_Y \theta = \sum (-1)^{i+1} (\mathcal{L}_Y \theta_i) \wedge \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \theta_n$$

so that

$$(p - \text{id}) \mathcal{L}_Y \theta = \sum (-1)^i (\mathcal{L}_Y \theta_i) (\bar{\partial}_j) \bar{\theta}_j \wedge \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \theta_n$$

and

$$p \mathcal{L}_X (p - \text{id}) \mathcal{L}_Y \theta = - \sum (\mathcal{L}_Y \theta_i) (\bar{\partial}_j) (\mathcal{L}_X \bar{\theta}_j) (\partial_i) \theta$$

The final result follows. \square

4.2 Metaplectic correction

Consider the set \mathcal{D} of linear map $D : \mathcal{C}^\infty(M, TM) \rightarrow \text{End}(\mathcal{C}^\infty(M, A))$ satisfying the following conditions

$$D_X(fs) = (X.f)s + f D_X s \quad (22)$$

$$D_{fX} s = f D_X s + \frac{1}{2} df(X^{1,0})s \quad (23)$$

for any vector field Y of M , function $f \in \mathcal{C}^\infty(M)$ and sections $s, t \in \mathcal{C}^\infty(M, A)$. In the case where X is a symplectic vector field, we also require that D_X is Hermitian,

$$X.(s, t) = (D_X s, t) + (s, D_X t), \quad s, t \in \mathcal{C}^\infty(M, A) \quad (24)$$

where $(s, t) \in \mathcal{C}^\infty(M)$ is the pointwise scalar product of s and t . Observe that for any $D \in \mathcal{D}$ and $\alpha \in \Omega^1(M, \text{Herm } A)$, $D + \frac{1}{i}\alpha$ belongs to \mathcal{D} .

Proposition 4.3. *The space \mathcal{D} is a real affine space directed by $\Omega^1(M, \text{Herm } A)$.*

Proof. The only difficulty is to check that \mathcal{D} is not empty. To do that, we will use the derivations D_X^K introduced in (21). Let ∇^A and ∇^K be Hermitian connections of A and K respectively. For any vector field X , define

$$D_X := \nabla_X^A + \frac{1}{2} B(X) \text{id}_A : \mathcal{C}^\infty(M, A) \rightarrow \mathcal{C}^\infty(M, A)$$

where $B(X) \in \mathcal{C}^\infty(M)$ is given by $B(X) = D_X^K - \nabla_X^K$. D_X is clearly a derivation in the direction of X . By Lemma 4.1, D_X satisfies Condition (23) and preserves the metric when X is symplectic. \square

Remark 4.4. Let δ be a half-form bundle, that is a Hermitian line bundle such that δ^2 is isomorphic to K . For any vector field X of M , let D_X^δ be the derivative of $\mathcal{C}^\infty(M, \delta)$ in the direction of X such that

$$D_X^K s^2 = 2s \otimes D_X^\delta s, \quad \forall s \in \mathcal{C}^\infty(M, \delta).$$

Write $A = B \otimes \delta$ where $B = A \otimes \delta^{-1}$. Then there is a one to one correspondence between the space of Hermitian connections of B and \mathcal{D} given as follows: for any Hermitian connection ∇^B of B , we set

$$D_Y = \nabla_Y^B \otimes \text{id} + \text{id} \otimes D_Y^\delta.$$

This provides another proof of Proposition 4.1 in the case where M has a half-form bundle. \square

Remark 4.5. Assume that A is a line bundle. Then there is a one to one correspondence between the space of Hermitian connections of $C = A^2 \otimes K^{-1}$ and \mathcal{D} given as follows: for any Hermitian connection ∇^C of C , we define first

$$D_X^{A^2} = \nabla_X^C \otimes \text{id} + \text{id} \otimes D_X^K$$

and then D_X is the unique derivation of $\mathcal{C}^\infty(M, A)$ with respect to X satisfying

$$D_X^{A^2}(s^2) = 2s \otimes D_X s, \quad \forall s \in \mathcal{C}^\infty(M, A). \quad \square$$

In the following proposition, we compute some kind of curvature for $D \in \mathcal{D}$.

Proposition 4.6. *For any $D \in \mathcal{D}$, there exists $R \in \Omega^2(M, \text{Herm } A)$ and a covariant derivation $\nabla : \mathcal{C}^\infty(M, \text{Herm } A) \rightarrow \Omega^1(M, \text{Herm } A)$ such that for any vector fields X, Y of M*

$$[D_X, D_Y] = D_{[X, Y]} + iR(X, Y) + \frac{1}{2}B_j(X, Y) \quad (25)$$

with $B_j(X, Y)$ the function defined in Lemma 4.2 and for any $f \in \mathcal{C}^\infty(M, \text{Herm } A)$ and $s \in \mathcal{C}^\infty(M, A)$,

$$D_X(f.s) = (\nabla_X f).s + f.D_X s. \quad (26)$$

Furthermore, the curvature of ∇ is $i \text{ad}_R$, $\nabla R = 0$, $\nabla \text{id} = 0$ and for any $f, g \in \mathcal{C}^\infty(M, \text{End } A)$, $\nabla(f.g) = (\nabla f).g + f.(\nabla g)$.

Proof. Assume first as in Remark 4.4 that $A = B \otimes \delta$ and $D_Y = \nabla_Y^B \otimes \text{id} + \text{id} \otimes D_Y^\delta$. Then we have a natural identification $\text{Herm } A \simeq \text{Herm } B$. We set $R = \frac{1}{i} \text{courb } \nabla^B$. Equation (25) follows from Lemma 4.2. We define ∇ as the connection of $\text{Herm } B$ induced by ∇^B , so that Equation (26) is satisfied and the properties of ∇ given in the proposition are standard properties.

We can now extend to the result to the case where there is no half-form bundle. Observe first that R and ∇ are uniquely determined by Equations (25) and (26). Since this unicity is also local, the local existence of R and ∇ implies their global existence. But each point of M has a neighborhood admitting a half-form bundle, where we can apply the first part of the proof. \square

5 The quantization map

Consider $D \in \mathcal{D}$. For any $f \in \mathcal{C}^\infty(M)$, define the derivative $P_{f,k}$ in the direction of the Hamiltonian vector field X of f

$$P_{f,k} = (\nabla_X^{L^k} \otimes \text{id} + \text{id} \otimes D_X) : \mathcal{C}^\infty(M, L^k \otimes A) \rightarrow \mathcal{C}^\infty(M, L^k \otimes A).$$

Then we set

$$Q_k^D(f) = \Pi_k(f + \frac{i}{k} P_{f,k}) : \mathcal{H}_k \rightarrow \mathcal{H}_k$$

Theorem 5.1. *Let $D \in \mathcal{D}$.*

1. *For any $f \in \mathcal{C}^\infty(M)$, $(Q_k^D(f))$ is a Toeplitz operator with principal symbol f .*
2. *For any f and $g \in \mathcal{C}^\infty(M)$, we have*

$$Q_k^D(f)Q_k^D(g) = Q_k^D(fg) + \frac{1}{2ik}\Pi_k\{f, g\} + \mathcal{O}(k^{-2})$$

and

$$ik[Q_k^D(f), Q_k^D(g)] = Q_k^D(\{f, g\}) + \frac{1}{k}\Pi_k R(X, Y) + \mathcal{O}(k^{-2}),$$

where X and Y are the Hamiltonian vector field of f and g respectively and $R \in \Omega^2(M, \text{Herm } A)$ is defined as in Proposition 4.6.

3. *For any $f \in \mathcal{C}^\infty(M)$ and Toeplitz operator S , $(ki[Q_k^D(f), S])$ is a Toeplitz operator with principal symbol*

$$\sigma_p(ki[Q_k^D(f), S]) = -\nabla_X(\sigma_p(S)).$$

where ∇ is the connection of $\text{Herm } A$ defined in Proposition 4.6 and X is the Hamiltonian vector field of f .

The theorem will be proved in Section 7. We can now defined a subprincipal symbol map $\sigma_s^D : \mathcal{T}_{\text{sc}} \rightarrow \mathcal{C}^\infty(M, \text{End } A)$ by

$$\sigma_s^D(T) := \sigma_p(k(T - Q^D(\sigma_p(T)))), \quad \forall T \in \mathcal{T}_{\text{sc}} \quad (27)$$

so that we have

$$T = Q^D(f) + k^{-1}\Pi g \text{ modulo } k^{-2}\mathcal{T} \quad \text{with } f = \sigma_p(T) \text{ and } g = \sigma_s^D(T). \quad (28)$$

Corollary 5.2. *For any $D \in \mathcal{D}$, σ_s^D belongs to \mathcal{E} . The two form R and the connection ∇ corresponding to σ_s^D , cf. Proposition 3.3, are the ones corresponding to D , cf. Proposition 4.6. Furthermore the map sending D into σ_s^D is an affine space isomorphism from \mathcal{D} to \mathcal{E} .*

Proof. σ_s^D clearly satisfies Conditions i) and ii). Condition iii) follows from the first equation of the second assertion of Theorem 5.1. The fact that the two form R and the covariant derivative ∇ corresponding to D and σ_s^D are the same follows from the second and third assertion of Theorem 5.1. Finally, recall that both \mathcal{D} and \mathcal{E} are affine spaces directed respectively by $\Omega^1(M, \text{Herm } A)$ and $\mathcal{C}^\infty(M, TM \otimes \text{Herm } A)$, cf. Propositions 3.2, 4.3. These vector spaces are isomorphic through ω . The map sending D to σ_s^D is a morphism of affine spaces. \square

6 Schwartz kernel of Toeplitz operators

The algebra \mathcal{A}_0

We briefly recall the definition and properties of the algebra \mathcal{A}_0 . The reader is referred to [Cha15] for a more detailed exposition. \mathcal{A}_0 depends on the data M, L, A, j . By definition, \mathcal{A}_0 consists of families $(P_k : \mathcal{C}^\infty(M, L^k \otimes A) \rightarrow \mathcal{C}^\infty(M, L^k \otimes A), k \in \mathbb{N}^*)$ of operators whose Schwartz kernels are smooth and of the following form: for any N , we have uniformly on M^2

$$P_k(x, y) = \left(\frac{k}{2\pi}\right)^n E^k(x, y) \sum_{\ell \in \mathbb{Z} \cap [-N, N/2]} k^{-\ell} f_\ell(x, y) + \mathcal{O}(k^{n-(N+1)/2}). \quad (29)$$

where E and the f_ℓ 's are sections of $E \boxtimes \overline{E}$ and $A \boxtimes \overline{A}$ respectively which satisfy the following conditions. For any $x, y \in M$, $E(x, x) = 1$, $|E(x, y)| < 1$ if $x \neq y$ and $\overline{E}(x, y) = E(y, x)$. For any $Z \in \mathcal{C}^\infty(M, T^{1,0}M)$, $\nabla_{(\overline{Z}, 0)} E$ vanishes to second order along the diagonal of M^2 . For any negative ℓ , f_ℓ vanishes to order -3ℓ along the diagonal. It is also required that the successive derivatives of $P_k(x, y)$ are uniformly slowly increasing as k tends to ∞ , cf. Section 2.2 of [Cha15] for a more precise formulation.

For any $m \in \mathbb{N}$, define \mathcal{A}_m as the subspace of \mathcal{A}_0 consisting of families with a Schwartz kernel in $\mathcal{O}(k^{n-m/2})$. By Theorem 3.3 of [Cha15], \mathcal{A}_0 is an algebra and $\mathcal{A}_m \cdot \mathcal{A}_p \subset \mathcal{A}_{m+p}$ for any m and p . Furthermore, we can describe the quotients $\mathcal{A}_m / \mathcal{A}_{m+1}$ by a convenient symbol and compute the corresponding products as follows. First $(P_k) \in \mathcal{A}_m$ if and only if in the asymptotic expansion (29), for any ℓ such that $-m \leq \ell \leq m/2$, the coefficient f_ℓ vanishes to order $m - 2\ell$ along the diagonal. If it is the case, the symbol of (P_k) is defined by

$$\sigma_m(P_k) = \sum_{\ell \in \mathbb{Z} \cap [-m, m/2]} \hbar^\ell [f_\ell]$$

where $[f_\ell] \in \mathcal{C}^\infty(M, S^{m-2\ell}(T^*M) \otimes \text{End } A)$ is the linearization of f_ℓ along the diagonal at order $m - 2\ell$. More explicitly, if $\partial_1, \dots, \partial_n$ is a local frame of $T^{1,0}M$ and (z_i) is the dual frame of $(T^{1,0}M)^*$, we set

$$[f_\ell](z, \overline{z}) = \sum_{|\alpha|+|\beta|=m-2\ell} \frac{1}{\alpha! \beta!} ((\overline{\partial}^\beta \boxtimes \partial^\alpha) f_\ell)|_{\text{diag } M} z^\alpha \overline{z}^\beta. \quad (30)$$

Then clearly $\sigma_m(P_k) = 0$ if and only if $(P_k) \in \mathcal{A}_{m+1}$. Furthermore if $(P_k) \in \mathcal{A}_p$ and $(Q_k) \in \mathcal{A}_q$ then the symbol of $(P_k Q_k) \in \mathcal{A}_{p+q}$ is equal to $\sigma_p(P) \star \sigma_q(Q)$ where \star is given by

$$(e \star g)(\hbar, z, \overline{z}) = \left[\exp(\hbar \Delta) (e(\hbar, -u, \overline{z} + \overline{u}) g(\hbar, z + u, -\overline{u})) \right]_{u=\overline{u}=0} \quad (31)$$

In this formula, $\Delta = \sum \partial_i \overline{\partial}_i$ acts on the variables u, \overline{u} .

Toeplitz operators

Denote by Π_k the orthogonal projector of $\mathcal{C}^\infty(M, L^k \otimes A)$ onto \mathcal{H}_k . Recall that the family (\mathcal{H}_k) is chosen in such a way that (Π_k) belongs to \mathcal{A}_0 and has symbol $\sigma_p(\Pi) = 1_A$. We have the following characterization of Toeplitz operators:

$$T \in \mathcal{T} \quad \Leftrightarrow \quad T \in \mathcal{A}_0 \text{ and } \Pi T \Pi = T.$$

This in particular gives a description of the Schwartz kernel of a Toeplitz operator. Furthermore, for any $T \in \mathcal{T}$ and $m \in \mathbb{N}$, $\sigma_{\text{cont}}(T) = \mathcal{O}(\hbar^m)$ if and only if $T \in \mathcal{A}_{2m}$. If it is the case, we have $\sigma_{2m}(T) = \hbar^m f$ and $\sigma_{\text{cont}}(T) = \hbar^m f + \mathcal{O}(\hbar^{m+1})$ for the same section $f \in \mathcal{C}^\infty(M, \text{End } A)$. Another useful property is that for any odd p , $\mathcal{A}_p \cap \mathcal{T} = \mathcal{A}_{p+1} \cap \mathcal{T}$.

Introduce a vector field X of M and a derivative $D_X : \mathcal{C}^\infty(M, A) \rightarrow \mathcal{C}^\infty(M, A)$ in the direction of X preserving the metric of A . Denote by $P_{X,k}$ the derivative

$$P_{X,k} = (\nabla_X^{L^k} \otimes \text{id} + \text{id} \otimes D_X) : \mathcal{C}^\infty(M, L^k \otimes A) \rightarrow \mathcal{C}^\infty(M, L^k \otimes A).$$

Lemma 6.1. *The operator family $(\frac{i}{k} P_{X,k} \Pi_k)$ belongs to \mathcal{A}_1 . Its symbol is the section $\tau_Y 1_A$ where $\tau_Y \in \mathcal{C}^\infty(M, T^*M)$ is given by $\tau_Y = \omega(\cdot, Y^{1,0})$.*

Proof. The Schwartz kernel of $\frac{i}{k} P_{X,k} \Pi_k$ being $(\frac{i}{k} P_{X,k} \boxtimes \text{id}) \Pi_k$, the result is a particular case of Lemma 2.19 in [Cha15]. \square

Lemma 6.2. *Assume that X is the Hamiltonian vector field of $f \in \mathcal{C}^\infty(M)$. Then the family $([f + \frac{i}{k} P_{X,k}, \Pi_k])$ belongs to \mathcal{A}_2 . Using the same notations as in Equation (30), its symbol is*

$$\left(\sum_{i,j=1}^n \omega(\bar{\partial}_i, [\bar{\partial}_j, X]) \bar{z}_i \bar{z}_j - \omega(\partial_i, [\partial_j, X]) z_i z_j \right) 1_A.$$

This is a particular case of Lemma 3.5 in [Cha15]. Let us deduce a first consequence of Lemma 6.1. Consider a second vector field Y of M and a derivation D_Y of $\mathcal{C}^\infty(M, A)$. Let $P_{Y,k}$ be the corresponding operator.

Lemma 6.3. *The operator family $(\frac{1}{k^2} \Pi_k P_{X,k} P_{Y,k} \Pi_k)$ belongs to \mathcal{A}_2 . Its symbol is $-i\hbar \omega(X^{0,1}, Y^{1,0})$.*

Proof. By Lemma 6.1, $(\frac{i}{k} P_{X,k} \Pi_k)$ belong to \mathcal{A}_1 and its symbol is $\tau_X 1_A$. Taking adjoint and using that $(\frac{i}{k} \Pi_k \text{div } X) \in \mathcal{A}_2$, we get that $(\Pi_k \frac{i}{k} P_{X,k})$ belongs to \mathcal{A}_1 . Furthermore its symbol is $\bar{\tau}_X 1_A$. Consequently, $(-\frac{1}{k^2} \Pi_k P_{X,k} P_{Y,k} \Pi_k)$ belongs to \mathcal{A}_2 . By Equation (31), its symbol is

$$\bar{\tau}_X 1_A \star \tau_Y 1_A = \hbar \sum \omega(\partial_i, X^{0,1}) \omega(\bar{\partial}_i, Y^{1,0})$$

where (∂_i) is an orthonormal frame of $T^{1,0}M$. Since $iX^{0,1} = \sum \omega(\partial_i, X^{0,1}) \bar{\partial}_i$, we obtain that $\bar{\tau}_X 1_A \star \tau_Y 1_A = i\hbar \omega(X^{0,1}, Y^{1,0})$. \square

7 Proof of Theorem 5.1

Consider $D \in \mathcal{D}$ as in section 4.2. Recall that for any function $f \in \mathcal{C}^\infty(M)$, $Q_k^D(f) = \Pi_k(f + \frac{i}{k}P_{f,k})\Pi_k$ where $P_{f,k} = \nabla_{X_f}^{L^k} \otimes \text{id} + \text{id} \otimes D_{X_f}$.

The fact that $Q_k^D(f)$ is a Toeplitz operator with principal symbol f was already observed in [Cha15]. Let us recall the proof since it is an easy consequence of Lemma 6.1. Since $(\frac{i}{k}P_{X,k}\Pi_k)$ belongs to \mathcal{A}_1 , the same holds for $(\Pi_k \frac{i}{k}P_{X,k}\Pi_k)$. So $(\Pi_k \frac{i}{k}P_{X,k}\Pi_k)$ is a Toeplitz operator in $\mathcal{T} \cap \mathcal{A}_1 = \mathcal{T} \cap \mathcal{A}_2$. Consequently

$$Q_k^D(f) = \Pi_k f \Pi_k \mod \mathcal{T} \cap \mathcal{A}_2,$$

so that $Q^D(f)$ is a Toeplitz operator with principal symbol f . This shows the first assertion of Theorem 5.1. The third assertion of Theorem 5.1 has already been proved in Theorem 5.8 of [Cha15]. It remains to prove the second assertion.

Lemma 7.1. *For any $f, g \in \mathcal{C}^\infty(M)$, we have*

$$\Pi_k(f + \frac{i}{k}P_{f,k})(g + \frac{i}{k}P_{g,k})\Pi_k = \Pi_k(fg + \frac{i}{k}P_{fg,k} + \frac{i}{2k}X.g)\Pi_k \mod \mathcal{A}_4 \cap \mathcal{T}.$$

with X the Hamiltonian vector field of f .

Proof. Let X and Y be the Hamiltonian vector fields of f and g respectively. Then the Hamiltonian vector field of fg is $fY + gX$. Using Condition (23), we obtain that

$$\begin{aligned} P_{fg,k} &= fP_{g,k} + gP_{f,k} + \frac{1}{2}(df(Y^{1,0}) + dg(X^{1,0})) \\ &= fP_{g,k} + gP_{f,k} + \frac{1}{2}(df(Y^{1,0}) + X.g - dg(X^{0,1})) \\ &= fP_{g,k} + gP_{f,k} + df(Y^{1,0}) + \frac{1}{2}X.g \end{aligned}$$

where we have used that

$$-dg(X^{0,1}) = \omega(X^{0,1}, Y) = \omega(X^{0,1}, Y^{1,0}) = \omega(X, Y^{1,0}) = df(Y^{1,0})$$

because $T^{1,0}M$ and $T^{0,1}M$ are Lagrangian. So we have on the one hand that

$$fg + \frac{i}{k}P_{fg,k} + \frac{i}{2k}X.g = fg + \frac{i}{k}(fP_{g,k} + gP_{f,k} + X.g) + \frac{i}{k}df(Y^{1,0}). \quad (32)$$

On the other hand, we have

$$(f + \frac{i}{k}P_{f,k})(g + \frac{i}{k}P_{g,k}) = fg + \frac{i}{k}(gP_{f,k} + fP_{g,k} + X.g) - \frac{1}{k^2}P_{f,k}P_{g,k} \quad (33)$$

Clearly $\Pi_k \frac{i}{k}df(Y^{1,0})\Pi_k$ belongs to \mathcal{A}_2 . Its symbol is $i\hbar df(Y^{1,0})$. By Lemma 6.3, $(-\Pi_k \frac{1}{k^2}P_{f,k}P_{g,k}\Pi_k) \in \mathcal{A}_2$ and has the same symbol. So

$$\Pi_k \frac{i}{k}df(Y^{1,0})\Pi_k = -\Pi_k \frac{1}{k^2}P_{f,k}P_{g,k}\Pi_k \mod \mathcal{A}_3.$$

Recall that $\mathcal{A}_3 \cap \mathcal{T} = \mathcal{A}_4 \cap \mathcal{T}$. So Equations (32) and (33) imply the result. \square

Theorem 7.2. *For any $f, g \in \mathcal{C}^\infty(M)$, we have*

$$Q_k^D(f)Q_k^D(g) \equiv Q_k^D(fg) + \frac{1}{2ki}Q_k^D(\{f, g\}) \pmod{\mathcal{A}_4 \cap \mathcal{T}}$$

where $\{f, g\}$ denotes the Poisson bracket of f and g .

Proof. Introduce the operators

$$S(f, g, k) = \Pi_k[f + \frac{i}{k}P_{f,k}, \Pi_k][g + \frac{i}{k}P_{g,k}, \Pi_k]\Pi_k$$

By Lemma 6.2, the family $(S(f, g, k))$ belongs to $\mathcal{A}_4 \cap \mathcal{T}$. A straightforward computations shows that

$$S(f, g, k) = Q_k^D(f)Q_k^D(g) - \Pi_k(f + \frac{i}{k}P_{f,k})(g + \frac{i}{k}P_{g,k})\Pi_k \quad (34)$$

So we have

$$\Pi_k(f + \frac{i}{k}P_{f,k})(g + \frac{i}{k}P_{g,k})\Pi_k = Q_k^D(f)Q_k^D(g) \pmod{\mathcal{A}_4 \cap \mathcal{T}}. \quad (35)$$

We conclude with Lemma 6.3. \square

Lemma 7.3. *For any $f, g \in \mathcal{C}^\infty(M)$, we have*

$$\begin{aligned} [f + \frac{i}{k}P_{f,k}, g + \frac{i}{k}P_{g,k}] &= \frac{1}{ik}(\{f, g\} + \frac{i}{k}P_{\{f, g\}, k}) \\ &\quad - \frac{1}{k^2} \text{id} \otimes (iR(X, Y) + \frac{1}{2}B_j(X, Y)) \end{aligned}$$

where R is the 2-form defined in Proposition 4.6, X and Y are the Hamiltonian vector fields of f and g , and $B_j(X, Y)$ is the function defined in Lemma 4.2.

Proof. A famous computation shows that

$$[f + \frac{i}{k}\nabla_X^{L^k}, g + \frac{i}{k}\nabla_Y^{L^k}] = \frac{i}{k}(-\{f, g\} + \frac{i}{k}\nabla_{[X, Y]}^{L^k})$$

where X and Y the Hamiltonian vector fields of f and g . The result is now a consequence of Proposition 4.6. \square

Lemma 7.4. *For any $f, g \in \mathcal{C}^\infty(M)$, $S(f, g, k) - S(g, f, k)$ belongs to \mathcal{A}_4 and its symbol is $\frac{\hbar^2}{2}B_j(X, Y)$.*

Proof. By Lemma 6.2, the symbol σ_X of $([f + \frac{i}{k}P_{f,k}, \Pi_k])$ has the form $f_X(z) - \overline{f_X(z)}$ where f_X is quadratic. Using this fact, a direct computation shows that

$$1_A \star \sigma_X \star \sigma_Y \star 1_A = \frac{\hbar^2}{2} \sum_{i,j} (\partial_i \partial_j \sigma_X) (\overline{\partial_i \partial_j \sigma_Y}).$$

where we denote by \star the product given in Equation (31). By Lemma 6.2,

$$(\partial_i \partial_j \sigma_X) = -\omega(\partial_i, [\partial_j, X]), \quad (\overline{\partial_i \partial_j \sigma_Y}) = \omega(\overline{\partial_i}, [\overline{\partial_j}, Y]).$$

Since (∂_i) is an orthonormal frame, the dual frame is given by $\theta_i = \frac{1}{i}\omega(\cdot, \bar{\partial}_i)$. Since Y preserves ω , we get $\mathcal{L}_Y\theta_i = \frac{1}{i}\omega(\cdot, [Y, \bar{\partial}_i])$. Replacing Y by X and taking the conjugate, $\mathcal{L}_X\bar{\theta}_i = -\frac{1}{i}\omega(\cdot, [X, \partial_i])$. Consequently

$$1_A \star \sigma_X \star \sigma_Y \star 1_A = -\frac{\hbar^2}{2} \sum_{i,j} (\mathcal{L}_X\bar{\theta}_i)(\partial_j)(\mathcal{L}_Y\theta_i)(\bar{\partial}_j)$$

Antisymmetrising, we get the final result. \square

Theorem 7.5. *For $f, g \in \mathcal{C}^\infty(M)$, we have that*

$$[Q_k^D(f), Q_k^D(g)] = \frac{1}{ik}(Q_k^D(\{f, g\}) + \frac{1}{k}\Pi_k R(X, Y)\Pi_k) \mod \mathcal{A}_6 \cap \mathcal{T}$$

where X and Y are the Hamiltonian vector fields of f and g .

Proof. By Equation (34),

$$S(f, g, k) - S(g, f, k) = [Q_k^D(f), Q_k^D(g)] - \Pi_k[f + \frac{i}{k}P_{f,k}, g + \frac{i}{k}P_{g,k}]\Pi_k$$

By Lemma 7.4, the left hand side is equal to $\frac{1}{k^2}\Pi_k\frac{1}{2}B_j(X, Y)\Pi_k$ modulo $\mathcal{A}_6 \cap \mathcal{T}$. By Lemma 7.3, the right hand side is equal to

$$[Q_k^D(f), Q_k^D(g)] - \frac{1}{ik}Q_k^D(\{f, g\}) + \frac{i}{k^2}\Pi_k R(X, Y)\Pi_k + \frac{1}{k^2}\Pi_k\frac{1}{2}B_j(X, Y)\Pi_k$$

modulo $\mathcal{A}_6 \cap \mathcal{T}$. The result follows. \square

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